

Asymptotic role of entanglement in quantum metrology

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Quantum systems allow one to sense physical parameters beyond the reach of classical statistics—with resolutions greater than $1/N$, where N is the number of constituent particles independently probing a parameter. In the canonical phase sensing scenario the *Heisenberg Limit* $1/N^2$ may be reached, which requires, as we show, both the *relative size of the largest entangled block* and the *geometric measure of entanglement* to be nonvanishing as $N \rightarrow \infty$. Yet, we also demonstrate that in the asymptotic N limit any precision scaling arbitrarily close to the Heisenberg Limit ($1/N^{2-\varepsilon}$ with any $\varepsilon > 0$) may be attained, even though the system gradually becomes noisier and separable, so that both the above entanglement quantifiers asymptotically vanish. Our work shows that sufficiently large quantum systems achieve nearly optimal resolutions despite their relative amount of entanglement being arbitrarily small. In deriving our results, we establish the *continuity relation of the quantum Fisher information* evaluated for a phaselike parameter, which lets us link it directly to the geometry of quantum states, and hence naturally to the geometric measure of entanglement.

I. INTRODUCTION

Quantum metrology is a vivid topic of research both at the theoretical and experimental levels [1–3]. With the help of quantum systems consisting of particles that independently sense a parameter of interest one may attain sensing resolutions beyond the reach of classical statistics—beyond the so-called *Standard Quantum Limit* (SQL) [4]. This limit states that the Mean-Squared Error (MSE) of estimation may at best scale inversely to the number of particles employed, i.e., as $1/N$. Quantum mechanics allows one to beat the SQL and in the canonical phase-sensing scenario reach a $1/N^2$ resolution—the *Heisenberg Limit* (HL)—a quantum enhancement of precision that limitlessly improves with N [5]. Spectacularly, quantum metrology schemes have been experimentally demonstrated to allow for enhanced sensing of phaselike parameters in optical interferometry [6], e.g., in gravitational-wave detection [7], but also in atomic-ensemble experiments of spectroscopy [8] and magnetometry [9], as well as in atomic clocks [10].

Quantum enhancement in metrology is only possible thanks to the *interparticle entanglement* exhibited by the quantum system employed [11]. In fact, resolutions beyond classical limits have been used to prove the existence of large-scale entanglement in real atomic systems [12]. The main obstacle in such experiments is the noise which destroys the entanglement and impairs the sensitivity [13]. For the attained precision to preserve the super-classical scaling, the number of entangled particles, or, formally, the entanglement *producibility* [14] (or *depth* [15]) must grow with the system size [16]. On the other hand, by studying the ultimate resolutions attainable with noisy quantum systems, it has been shown that generic uncorrelated (independently disturbing the particles) noise-types limit the quantum enhancement to a constant factor [17]. In terms of entanglement properties, such SQL-like sensitivities can then be reached for arbitrary large N by grouping the constituent particles into separate entangled blocks of finite size [18]. Although the protection of entanglement is thus

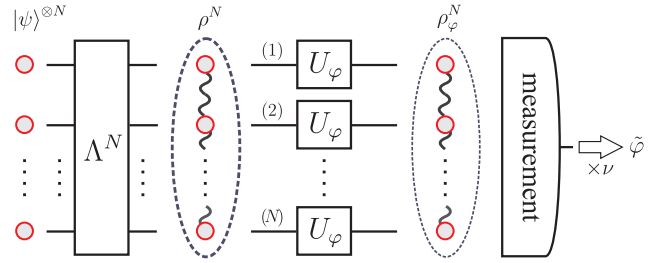


FIG. 1. **Quantum phase sensing protocol**—designed to most precisely sense fluctuations of a phase-like parameter φ around its given value. The system is prepared in an N -particle entangled state ρ^N obtained from $|\psi\rangle^{\otimes N}$ by the preparation map Λ^N that also incorporates noise. φ is encoded on each particle by a unitary U_φ and the final state ρ_φ^N is measured. The procedure is repeated sufficiently many times ($\nu \gg 1$) to construct most accurate parameter estimate $\hat{\varphi}$.

of the highest priority for the super-classical precision scaling to be preserved, there also exist states that possess all their particles (genuinely) entangled but nonetheless are useless for metrology [19]. Moreover, the nature of entanglement that is essential for metrological purposes remains unclear, as by employing large-scale but yet undistillable entanglement (which could be considered of the weakest type [20]) one may still attain the HL resolution in phase sensing [21].

In this work, we connect the key metrological performance quantifier—the asymptotic scaling of precision—with the entanglement properties as quantified relatively to the size of the system employed. To this end, we first establish a *continuity relation for the quantum Fisher information* (QFI), which allows us to connect the metrological properties of quantum states to their geometry. Thanks to the derived continuity, we are able to upper bound the QFI by the geometric measure of entanglement (GME) [22]. As a result, we demonstrate that, although to attain the *exact* HL both the relative size of the largest entangled block and the GME must be asymptotically nonvanishing, any precision scaling *arbitrarily close* to HL, $1/N^{2-\varepsilon}$ with $\varepsilon > 0$, is achievable despite both these entanglement quantifiers decaying with $N \rightarrow \infty$.

II. PRELIMINARIES

A. Metrology protocol

We consider the noisy phase sensing protocol depicted in Fig. 1, which allows us to unambiguously approach the problem. We encode the parameter unitarily, so that the asymptotic precision scaling can be actually improved [17], and independently on each particle—so that the quantum-enhanced scaling is firmly constrained between SQL and HL ($1/N$ and $1/N^2$) and attributed solely to the entanglement properties of the quantum state of the system [23]. On the other hand, for any entanglement quantifier to be comparable with the asymptotic precision scaling it must be “scale-independent”, i.e., it cannot grow with N when considering a sequence of states of the same type [24]. Hence, a notion of entanglement “size” may only be quantified relatively to the total system size, while the entanglement “amount” must not change by simply increasing N . We define adequately both such notions below, but let us already stress that the latter we find to be naturally emergent by relating the metrological properties of quantum states to their geometry.

We follow the frequentist approach to estimation which applies in the regime of sufficiently many independent protocol repetitions ($\nu \gg 1$ in Fig. 1), while sensing small parameter fluctuations around its certain known value [25]. Then, the MSE, $\Delta^2 \tilde{\varphi}$, of any (consistent and unbiased) estimator, $\tilde{\varphi}$, of the parameter is ultimately lower limited by the *Quantum Cramér-Rao Bound* [26]:

$$\Delta^2 \tilde{\varphi} \geq \frac{1}{\nu F_Q[\rho_\varphi^N]}, \text{ where } F_Q[\rho_\varphi^N] := \text{Tr}\{\rho_\varphi^N L[\rho_\varphi^N]^2\} \quad (1)$$

is the *Quantum Fisher Information* (QFI) for a given N -particle state ρ_φ^N with φ standing for the true parameter value. $L[\rho_\varphi^N]$ is the symmetric logarithmic derivative operator unambiguously defined via $\partial_\varphi \rho_\varphi^N = (L[\rho_\varphi^N] \rho_\varphi^N + \rho_\varphi^N L[\rho_\varphi^N])/2$ [27].

In the customary phase sensing protocol of Fig. 1 the estimated parameter is encoded onto the system state ρ^N via $\rho_\varphi^N = U_\varphi^{\otimes N} \rho^N U_\varphi^{\dagger \otimes N}$ with $U_\varphi = e^{-ih\varphi}$ and h being some fixed single-particle Hamiltonian. Without loss of generality we assume the operator norm of h to fulfil $\|h\| \leq 1/2$, so that the single-particle QFI generally satisfies $F_Q[\rho_\varphi^1] \leq 1$. As the parameter encoding is unitary, in what follows we may write the QFI for given h as $F_Q[\rho^N] := F_Q[\rho_\varphi^N]$ manifesting its independence of φ [3]. Moreover, as the QFI is additive and convex [3], it must then fulfil $F_Q[\rho_{\text{sep}}^N] \leq N$ for any separable ρ_{sep}^N . Hence, this proves that the SQL can be surpassed indeed *only* when the quantum state ρ^N exhibits entanglement [11].

B. Entanglement quantifiers

In order to quantify the *relative size* of entanglement contained in a given ρ^N of Fig. 1, we use the notion of *producibility* [14] (also termed *entanglement depth* [15]). An N -particle pure state is termed *k-producible* if it can be written as $|\psi^N\rangle = \otimes_{m=1}^{M \leq N} |\psi_m\rangle$ with each $|\psi_m\rangle$ consisting of *at most* k

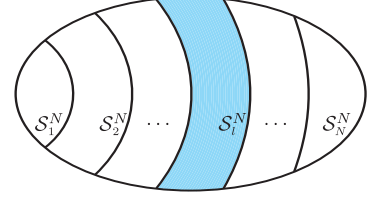


FIG. 2. **Hierarchy of convex sets \mathcal{S}_k^N containing all k -producible states.** The set \mathcal{S}_N^N contains all states acting on $\mathcal{H}^{\otimes N}$, while \mathcal{S}_1^N is its subset of separable states. Shaded region is the set $\mathcal{S}_l^N \setminus \mathcal{S}_{l-1}^N$ of states with the relative size of largest entangled block: $R_{\text{LEB}} = l/N$.

particles. This directly extends to mixed states: a mixed state ρ^N is *k-producible* if it is a convex combination of pure *k-producible* states [14]. Hence, for an N -particle Hilbert space $\mathcal{H}^{\otimes N}$, the *convex* sets of all *k-producible* states, which we denote by \mathcal{S}_k^N , form a hierarchy $[\mathcal{S}_1^N \subset \mathcal{S}_2^N \subset \dots \subset \mathcal{S}_N^N]$ that we schematically depict in Fig. 2. Note that \mathcal{S}_1^N is just the set of fully separable states, \mathcal{S}_N^N is the set of all states acting on $\mathcal{H}^{\otimes N}$, while $\mathcal{S}_N^N \setminus \mathcal{S}_{N-1}^N$ contains ones that are genuinely entangled—they do not admit any form of separability. Crucially, the concept of producibility allows us to define for any N -particle state $\rho_{(l)}^N$ that is *l-producible* but *not* $(l-1)$ -producible, i.e., $\rho_{(l)}^N \in \mathcal{S}_{(l)}^N := \mathcal{S}_l^N \setminus \mathcal{S}_{l-1}^N$, the *relative size of Largest Entangled Block (LEB)* of particles as $R_{\text{LEB}} := l/N$. Thus, R_{LEB} is the ratio of the size of the largest subgroup of particles that are entangled to the total particle number. When describing the precision scaling attained by metrology protocols, one deals with the $N \rightarrow \infty$ limit. Note that in such an asymptotic regime the LEB may be divergent despite R_{LEB} vanishing with N . Hence, if one was to associate the entanglement size with the number of particles being entangled via LEB, in many situations it would be infinite for $N \rightarrow \infty$. In contrast, R_{LEB} adequately then takes values within the interval $[0, 1]$ depending on the sequence of states considered.

On the other hand, in order to quantify the amount of entanglement exhibited by ρ^N in Fig. 1, we employ the *geometric measure of entanglement* (GME) that is defined for pure states as $E_G[|\psi^N\rangle] := 1 - \max_{|\phi^N\rangle \in \mathcal{S}_1^N} |\langle \phi^N | \psi^N \rangle|^2$ [22]. Its definition naturally generalises to mixed states through the convex-roof construction [22]:

$$E_G[\rho^N] := \inf_{\{p_i, |\psi_i^N\rangle\}} \sum_i p_i E_G[|\psi_i^N\rangle] \quad (2)$$

with the infimum taken over all ensembles $\{p_i, |\psi_i^N\rangle\}$ such that $\rho^N = \sum_i p_i |\psi_i^N\rangle \langle \psi_i^N|$. However, one may show that definition (2) may be equivalently obtained by employing the Uhlmann fidelity, $F(\rho, \sigma) := \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$ [28], so that $E_G[\rho^N] = 1 - \max_{\sigma^N \in \mathcal{S}_1^N} F^2(\rho^N, \sigma^N)$ [29]. Crucially, thanks to its geometrical formulation, the GME is independent of the particle number N . In particular, it effectively measures the distance to separable states independently of the Hilbert space dimension. To see this, note that the GME obeys the following inequality:

$$1 - \sqrt{1 - E_G(\rho^N)} \leq \min_{\sigma^N \in \mathcal{S}_1^N} T(\rho^N, \sigma^N) \leq \sqrt{E_G(\rho^N)}, \quad (3)$$

which directly follows from the Fuchs-van de Graaf relation between fidelity and trace-distance, $T(\rho, \sigma) := \|\rho - \sigma\|_1/2$, of quantum states [30]. In particular, for any sequence of states $\{\rho^N\}$ that is bounded away from the set of fully separable states, in the sense that $\min_{\sigma^N \in \mathcal{S}^N} T(\rho^N, \sigma^N) \geq c$ with some constant $c > 0$, the GME is also bounded away from zero as $E_G(\rho^N) \geq c^2$. On the other hand, the vanishing GME of a sequence $\{\rho^N\}$ implies that its elements must converge to the set of separable states as $N \rightarrow \infty$. As a result, the GME may have been used to demonstrate, e.g., that typical states—by exhibiting high GME—have high entanglement [31].

III. RESULTS

A. Continuity of QFI

Our first result is the continuity relation for the QFI, which will later allow us to naturally connect the GME of a state with its metrological properties. More precisely, exploiting the *purifications-based definition* of QFI [17], we upper bound in App. A the difference of QFIs for any two quantum states via their geometrical separation, in particular, via their *fidelity*, *trace* or *Bures distance*. In the special case of one of the states being pure, we additionally tighten the corresponding bound utilising the *convex-roof-based definition* of QFI [32]. The result may be summarised by the following inequality holding for any two $\rho^N, \sigma^N \in \mathcal{B}(\mathcal{H}^{\otimes N})$ (see App. A for the proof):

$$|F_Q[\rho^N] - F_Q[\sigma^N]| \leq \xi \sqrt{1 - F(\rho^N, \sigma^N)^2} N^2, \quad (4)$$

where $F(\rho, \sigma)$ is again the Uhlmann fidelity, while $\xi = 8$ for general quantum states and $\xi = 6$ if one of them is pure.

Let us first stress the general power of the continuity relation (4) when used for comparing metrological properties between multipartite states. It straightforwardly follows from Eq. (4) that for any pair ρ^N, σ^N (see also App. A):

$$F_Q[\rho^N] \leq F_Q[\sigma^N] + \xi \sqrt{2 T(\rho^N, \sigma^N)^2} N^2 \quad (5)$$

with $T(\rho, \sigma)$ denoting the trace distance as before. Hence, Eq. (5) directly implies that given a sequence of states $\{\sigma^N\}$ that *do not attain* the HL, i.e., $F_Q[\sigma^N]/N^2 \rightarrow 0$ with N , *any* other sequence $\{\rho^N\}$ which consists of states those successively converge to $\{\sigma^N\}$, so that $T(\rho^N, \sigma^N) \rightarrow 0$ with N , *cannot* attain the HL either. In particular, recalling that separable states do not allow for any quantum-enhanced sensitivity, no sequence of states tending to the set of fully separable states \mathcal{S}_1^N may attain the HL. On the other hand, taking in contrast $\{\rho^N\}$ in Eq. (5) as the reference sequence that *attains* the HL, i.e., $F_Q[\rho^N]/N^2 \rightarrow c > 0$ with N , Eq. (5) proves that *any* other sequence $\{\sigma^N\}$ *must also* attain the HL, as long as $\xi \sqrt{2 T(\rho^N, \sigma^N)} \rightarrow c' < c$ while $N \rightarrow \infty$. For instance, choosing as reference the optimal GHZ states, which yield $F_Q[\psi_{\text{GHZ}}^N] = N^2$ and, hence, $c = 1$ ($\xi = 6$), we see that any other sequence of states $\{\sigma^N\}$ that maintain their $T(\psi_{\text{GHZ}}^N, \sigma^N) < 1/72$ with N must also follow the HL. This is consistent with recent profound methods which focus on

Dicke-state sequences and imply $T(\psi_{\text{GHZ}}^N, \sigma^N) < \sqrt{3}/2$ to be sufficient in case of GHZ states [33].

Surprisingly, Eq. (5) opens an interesting possibility: there may exist two sequences of states which asymptotically converge despite contrasting metrological properties. Consider a sequence of states $\{\sigma^N\}$ such that $F_Q[\sigma^N] \sim N^\alpha$ for sufficiently large N , yielding a $1/N^\alpha$ asymptotic resolution with $0 < \alpha < 2$ (possibly even sub-SQL). Eq. (5) does not exclude the existence of another sequence $\{\rho^N\}$ with $T(\rho^N, \sigma^N) \rightarrow 0$ for $N \rightarrow \infty$ that nonetheless attains *any* improved precision scaling $1/N^{2-\varepsilon}$ with $0 < \varepsilon < 2 - \alpha$. All what Eq. (5) imposes is that $\{\rho^N\}$ approaches $\{\sigma^N\}$ slow enough, so that $T(\rho^N, \sigma^N) \gtrsim 1/N^{2\varepsilon}$ as $N \rightarrow \infty$. In the context of entanglement, there may thus exist sequences approaching the set of *separable* states but preserving precision scaling arbitrarily close to HL. We later provide examples of such sequences.

B. Relating QFI to entanglement

We first recall the result of [16] relating the notions of QFI and k -producibility: for any k -producible state $\sigma^N \in \mathcal{S}_k^N$, the QFI is upper bounded as

$$F_Q[\sigma^N] \leq \left\lfloor \frac{N}{k} \right\rfloor k^2 + \left(N - \left\lfloor \frac{N}{k} \right\rfloor k \right)^2 \leq kN, \quad (6)$$

where $\lfloor x \rfloor := \text{floor}[x]$. The above bound importantly implies that for states with fixed producibility k (independent of N) the quantum enhancement is limited to a constant factor [16]. Hence, for a super-classical precision scaling to be possible the preparation map Λ^N in Fig. 1 must output states such that their producibility constantly rises with increasing N .

On the other hand, in terms of R_{LEB} , Eq. (6) equivalently reads: $F_Q[\sigma^N] \leq R_{\text{LEB}} N^2$. Thus, the *exact* HL can be attained *only if* R_{LEB} does not vanish in the asymptotic N limit, what requires the relative size of entanglement to be maintained with increasing N . However, similarly to the continuity relation (5), Eq. (6) leaves open the existence of sequences attaining scalings *arbitrarily close* to HL despite their R_{LEB} tending to zero with N (it requires the size of the particle LEB to grow as $N^{1-\varepsilon}$, letting R_{LEB} vanish as $N^{-\varepsilon}$). Operationally, in order to reach a super-classical scaling, it is thus enough for the Λ^N of Fig. 1 to prepare states with the effective number of entangled particles rising with N , yet at such a rate that its ratio to the total particle number is constantly decreasing. One may thus argue that the preparation map Λ^N of Fig. 1 is then experimentally easier to implement, as it does not require the relative size of entanglement to be maintained with increasing N (e.g., while squeezing an atomic ensemble [34]), especially when dealing with systems of macroscopic size [35].

Let us now provide the second main result relating the QFI and the GME. To this end, we show that Eq. (4) (with $\xi = 6$) may be utilised to upper bound the QFI as (see App. B for the proof):

$$F_Q[\rho^N] \leq N + 6 \sqrt{E_G[\rho^N]} N^2. \quad (7)$$

As an aside, note that the formula (7) may be straightforwardly generalised to any $k > 1$ with help of the bound (6) and by defining the *geometric measure of k -producibility* after replacing S_1^N with S_k^N in Eq. (2) (see App. B).

Inequality (7) implies that the *exact* HL can *only* be attained if the entanglement is asymptotically nonvanishing, i.e., any sequence $\{\rho^N\}$ with GME vanishing for $N \rightarrow \infty$ *cannot* reach the $1/N^2$ scaling. Still, Eq. (7) does not exclude the possibility that any resolution *arbitrarily close* to HL is attained by a sequence $\{\rho^N\}$, whose elements exhibit vanishingly small geometric measure of entanglement as $N \rightarrow \infty$. In particular, Eq. (7) just requires the GME to decay slowly enough, so that as long as asymptotically $E_G[\rho^N] \gtrsim 1/N^{2\varepsilon}$, any resolution $1/N^{2-\varepsilon}$ is allowed.

C. Almost the HL with vanishing R_{LEB} and GME

In order to affirm the above claims, we now provide examples of state sequences—consisting of either pure or mixed states—that attain precision scalings arbitrarily close to HL despite their relative size and amount of entanglement, as quantified by R_{LEB} and GME respectively, vanishing with $N \rightarrow \infty$. We return to the phase sensing scenario of Fig. 1 with the parameter φ being unitarily encoded via the single-particle Hamiltonian $h = \sigma_z/2$.

First, let us consider non-maximally entangled states:

$$|\psi_p^N\rangle := \sqrt{p}|0\rangle^{\otimes N} + \sqrt{1-p}|1\rangle^{\otimes N} \quad (8)$$

with $0 \leq p \leq 1/2$, so that $|\psi_{1/2}^N\rangle$ is the GHZ state of N qubits. The metrological capabilities of states (8) were studied in Ref. [19], where it was shown that by making p vanish quickly enough with N , states (8) do not surpass SQL despite being genuinely entangled for any $p > 0$. On the contrary, we focus on the fact that states (8) also allow for resolutions arbitrarily close to HL even when $p \rightarrow 0$ as $N \rightarrow \infty$. However, in order to also control and vary their LEB, we tailor them to $|\psi_{p,l}^N\rangle := |\psi_p^N\rangle \otimes |0\rangle^{\otimes N-l}$, so that their $R_{\text{LEB}} = l/N$ for any $p > 0$. As $E_G[|\psi_{p,l}^N\rangle] = p$ [22], we may then rewrite their QFI as $F_Q[|\psi_{p,l}^N\rangle] = 4p(1-p)l^2 = 4E_G(1-E_G)R_{\text{LEB}}^2N^2$. Thus, by setting both $E_G = 1/N^{\varepsilon_1}$ and $R_{\text{LEB}} = 1/N^{\varepsilon_2}$ to vanish with N for any $\varepsilon_1, \varepsilon_2 > 0$, we obtain the QFI to scale as $F_Q[|\psi_{p,l}^N\rangle] \sim N^{2-\varepsilon_1-2\varepsilon_2}$, which in turn yields the desired arbitrarily close to HL resolution $1/N^{2-\varepsilon_1-2\varepsilon_2}$.

Now, let us turn to the case of mixed states and consider N -qubit Werner-type states [36]:

$$\rho_p^N = p|\psi_{1/2}^N\rangle\langle\psi_{1/2}^N| + (1-p)\frac{\mathbb{1}_{2^N}}{2^N}. \quad (9)$$

The QFI of ρ_p^N reads $F_Q[\rho_p^N] = N^2p^2/[p + (1-p)/2^{N-1}]$ [3], and for sufficiently large N simplifies to pN^2 (independently whether p depends on N). Although the GME can be exactly evaluated for these states [37], for our purposes it is enough to use the upper bound $E_G[\rho_p^N] \leq p/2$, which stems from the convexity of GME and may be shown to be saturated for $N \rightarrow \infty$ (see App. C). Thus, for sufficiently large N we may write

$F_Q[\rho_p^N] \geq 2E_GN^2$. Note that by setting $p = 1/N^\varepsilon$, leading to $E_G[\rho_p^N] \leq 1/(2N^\varepsilon)$, we actually let the white noise increase with N , so that the state (9) becomes fully depolarised in the asymptotic N limit. Nevertheless, the QFI scales then at least as $F_Q[\rho_p^N] \gtrsim N^{2-\varepsilon}$ leading to the claimed $1/N^{2-\varepsilon}$ resolution. Moreover, it has been proven that for states (9) to be genuinely entangled $p > (2^{N-1} - 1)/(2^N - 1)$ [38], which for large N converges to $1/2$ from above. Hence, by letting $p \rightarrow 0$ as $N \rightarrow \infty$ we obtain a sequence of states that quickly seize to be genuinely entangled with strictly $R_{\text{LEB}} < 1$. However, in order to prove that R_{LEB} can be made vanishing, similar to the pure states case, we tailor the states (9) accordingly to

$$\rho_{p,l}^N := p|\psi_{1/2,l}^N\rangle\langle\psi_{1/2,l}^N| + (1-p)\frac{\mathbb{1}_{2^N}}{2^N}, \quad (10)$$

so that $R_{\text{LEB}} \leq l/N$ may be assured. Following the same argumentation as in the case of Eq. (9) [3], the QFI of states (10) can then be shown to simplify to $F_Q[\rho_{p,l}^N] \approx pl^2$ for sufficiently large N . Hence, as the GME has to still obey $E_G[\rho_{p,l}^N] \leq p/2$, the QFI of states (10) must asymptotically scale at least as $F_Q[\rho_{p,l}^N] \gtrsim 2E_GR_{\text{LEB}}^2N^2$. Thus, as desired, also the mixed states (10) allow us to set both $E_G = 1/N^{\varepsilon_1}$ and $R_{\text{LEB}} = 1/N^{\varepsilon_2}$ vanishing, but still attain the $1/N^{2-\varepsilon_1-2\varepsilon_2}$ resolution despite becoming completely depolarised in the asymptotic N limit.

Although the above pure- and mixed-state sequences demonstrate that, indeed, both the GME and R_{LEB} may be set vanishing as $N \rightarrow \infty$, while maintaining the arbitrarily close to HL resolutions, the exemplary sequences do not asymptotically saturate the bounds on the QFI set by Eqs. (6) and (7). In the latter case, we expect this to be a consequence of the QFI continuity relation (4) actually not being asymptotically saturable due to the “square-root” dependence on the distance between quantum states appearing in its form.

To put our results on firm ground, let us assume that one wants to attain a super-classical resolution that is close to HL, e.g., $1/N^{1.7}$. Then, in the case of pure (8) and mixed (10) states it may be reached after letting both the R_{LEB} and GME vanish with $\varepsilon_1 = \varepsilon_2 = 0.1$. Hence, when considering the large-particle-number regime of $N \approx 10^6$ (typical to atomic-ensemble experiments [9, 10]), one requires $E_G \approx 0.25$, which is half the entanglement of the GHZ state, and $R_{\text{LEB}} \approx 25\%$, that is, one-fourth of particles need to be entangled.

D. Geometric interpretation of the results

In Fig. 3, we schematically present an exemplary path that elements of sequences $\{\sigma_{(l)}^N\}$ and $\{\rho^N\}$ should take for the above described phenomenon to be possible: despite becoming arbitrarily close to each other as $N \rightarrow \infty$, the states $\{\sigma_{(l)}^N\}$ and $\{\rho^N\}$ have drastically different metrological properties. To be more precise, let the states $\sigma_{(l)}^N \in \mathcal{S}_{(l)}^N$ be of constant $\text{LEB} = l$ for all N . According to Eq. (6), their QFI is thus always constrained by lN , so that they only may yield an SQL-like precision scaling. On the other hand, let ρ^N be states whose LEB grows with N in a way that they attain an asymptotic precision scaling arbitrarily close to HL. Crucially, the

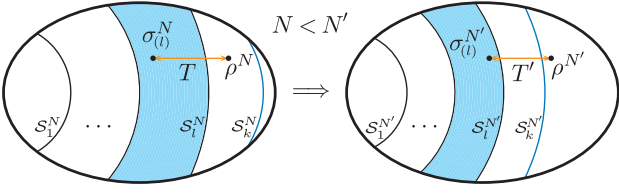


FIG. 3. **Collapse of the hierarchy of sets S_k^N** allowing two sequences $\{\sigma_{(l)}^N\}$ and $\{\rho^N\}$ to approach one another with N , despite their contrasting metrological properties. States $\sigma_{(l)}^N$ are of constant $\text{LEB} = l$, which (due to Eq. (6)) constrains their QFI to scale at most linearly with N : $F_Q[\sigma_{(l)}^N] \leq lN$. ρ^N are states with their LEB rising with N , so that their QFI is taken to scale as N^α with some $\alpha < 2$. Still, it is possible to choose the states $\sigma_{(l)}^N$ and ρ^N (e.g., ones of Eq. (10)) so that for sufficiently large N they become arbitrarily close to each other. In particular, for the two $N' > N$ depicted above, both $\sigma_{(l)}^N \in S_l^N$ and $\sigma_{(l)}^{N'} \in S_l^{N'}$, whereas $\rho^N \in S_k^N$ but $\rho^{N'} \notin S_k^{N'}$ for some $k > l$, although the trace distance ($T' < T$) between $\sigma_{(l)}^N$ and ρ^N decreases with N . This is possible because the sets S_l^N collapse faster than ρ^N approach $\sigma_{(l)}^N$.

two sequences exhibit highly contrasting metrological properties in the asymptotic N limit. Still, it is possible to choose them in a way that the geometric distance between their consecutive elements gradually vanishes as $N \rightarrow \infty$. As shown in Fig. 3, this is possible as the elements of $\{\rho^N\}$ are constantly “overtaken” by the boundaries of sets of higher producibility, while the hierarchy collapses with increasing N . We explicitly draw the boundaries of the k -producible sets for particle numbers $N < N'$, in order to emphasize that our results suggest rapid shrinkage of the sets with N . In particular, note that in Fig. 3: $\rho^N \in S_k^N$ but $\rho^{N'} \notin S_k^{N'}$; even though $T' = T(\rho^{N'}, \sigma_{(l)}^{N'}) < T = T(\rho^N, \sigma_{(l)}^N)$. We expect such a phenomenon to be the consequence of the volume of each S_k^N collapsing exponentially with N , which, according to our best knowledge has only been proven for the set of separable states (i.e., for $l = 1$) [39].

IV. CONCLUSION

We have studied restrictions that entanglement features impose on the asymptotic metrological performance of quantum states. First, by establishing the continuity relation for the QFI, we have related the metrological properties of states to their underlying geometry. This allowed us to naturally link their metrological utility to their entanglement content as measured by the geometric measure of entanglement. As a result, we have shown that for the HL to be attained in the asymptotic N limit both the relative size and amount of entanglement (as quantified by R_{LEB} and GME respectively) cannot vanish. For instance, the states that exhibit undistillable entanglement, but still attain the exact HL [21], must thus asymptotically possess finite R_{LEB} and GME. On the contrary, we have demonstrated that any precision scaling arbitrarily close to HL may be reached even though both R_{LEB} and GME vanish as $N \rightarrow \infty$. In the presence of global depolarisation, this still allows the decoherence strength to be increasing

with N , which contrasts the case of uncorrelated noise-types whose strength must decrease with system size for a scaling quantum-enhancement to be observed [17]. As uncorrelated noises yield $R_{\text{LEB}} \sim 1/N$ in the asymptotic N limit [18], our results provide a new (entanglement-degrading) interpretation of their destructive impact. We hope that our work can thus be beneficial in proposing novel noise-robust metrology schemes that attain super-classical resolutions by employing quantum states with just the necessary entanglement properties. As the metrological usefulness of a quantum state is directly related to its *macroscopicity* [40], all our results also apply in this context. Let us finally notice that our continuity of the QFI has recently been used by some of us to study the typical metrological properties of various ensembles of quantum states [41].

A natural open question to ask is how the conclusions of our work vary if one considers single-shot ($\nu = 1$ in Fig. 1) metrology protocols, in which the estimated parameter may not be assumed to be fluctuating around a known value and the Bayesian approach to estimation must be pursued [42]. Yet, we expect the requirements on the entanglement to be then much more stringent due to the lack of sufficient statistics.

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Appendix A: Continuity of QFI

Here we present a detailed proof of the continuity relation for the quantum Fisher information (QFI), which we extensively use in the main text. Yet, in order to also establish a common notation and preliminary notions, we firstly introduce the basic concepts of: a *purification of a mixed state*, and that of the *Uhlmann fidelity* and the *Bures distance* [28, 43].

Let us consider a quantum system represented by a mixed state ρ acting on a Hilbert space $\mathcal{H}_S = \mathbb{C}^d$. It follows that ρ can always be represented by a pure state from a larger Hilbert space. Concretely, there exists $|\psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E = \mathbb{C}^d \otimes \mathbb{C}^{d'}$ with $d' = \text{rank}(\rho)$ such that $\rho = \text{Tr}_E |\psi\rangle\langle\psi|$. This representation is, however, not unique because any pure state related to $|\psi\rangle$ via $|\psi'\rangle = \mathbb{1}_S \otimes V_E |\psi\rangle$ with V_E being some partial isometry ($V_E^\dagger V_E = \mathbb{1}$) is also a *purification* of ρ . At this point it is important to mention that any such V_E can be extended to

a unitary operation by properly enlarging the “environmental” Hilbert space \mathcal{H}_E , and so any two purifications of a given ρ are thus related by a unitary operation acting on \mathcal{H}_E [43].

Then, the *Uhlmann fidelity* of a pair of density matrices ρ and σ acting on $\mathcal{H}_S = \mathbb{C}^d$ is defined through

$$F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1 = \text{Tr}\sqrt{\sigma^{1/2}\rho\sigma^{1/2}}, \quad (\text{A1})$$

where $\|\cdot\|_1$ stands for the trace norm defined as $\|X\|_1 = \text{Tr}\sqrt{X^\dagger X}$. If at least one of these two states is pure, say $\sigma = |\phi\rangle\langle\phi|$, then the above formula simplifies to $F(\rho, |\phi\rangle) = \sqrt{\langle\phi|\rho|\phi\rangle}$, and if also $\rho = |\psi\rangle\langle\psi|$ then $F(|\psi\rangle, |\phi\rangle) = |\langle\psi|\phi\rangle|$ is just the overlap of the two pure states.

For further benefits let us also mention that for any pair of mixed states ρ and σ their fidelity can be expressed in terms of the fidelity of their purifications, denoted by $|\psi\rangle$ and $|\phi\rangle$, respectively. More precisely,

$$F(\rho, \sigma) = \max_{|\psi\rangle} F(|\psi\rangle, |\phi\rangle) = \max_{|\psi\rangle} |\langle\psi|\phi\rangle| \quad (\text{A2})$$

where the maximization is performed over all purifications of ρ , but equally well can be performed over the purifications of σ [43]. The Uhlmann fidelity does not fulfil properties of a *measure of distance* between quantum states [28], yet with its help one may define the so-called *Bures distance* [28]:

$$D_B(\rho, \sigma) = \sqrt{2[1 - F(\rho, \sigma)]}. \quad (\text{A3})$$

Having these notions at hand, we can now pass to the continuity relations of the QFI. Let us first recall that in our case the parameter φ is encoded on a state with the aid of a unitary operation, so that $\rho_\varphi = U_\varphi \rho U_\varphi^\dagger$ where $U_\varphi = e^{-iH\varphi}$ and H is a given parameter-encoding Hamiltonian. In such case, the QFI most generally reads

$$F_Q[\rho; H] := F_Q[\rho_\varphi] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle\xi_k|H|\xi_l\rangle|^2, \quad (\text{A4})$$

where λ_k and $|\xi_k\rangle$ are respectively the eigenvalues and the eigenvectors of ρ . Importantly, as emphasised by our notation in the definition (A4), owing to the unitary parameter-encoding, the QFI is independent of the estimated parameter and thus becomes just a function of the state and the Hamiltonian.

Theorem 1. *For any pair of density matrices ρ and σ acting on \mathcal{H}_S and for the QFI given in Eq. (A4) the following inequalities hold true:*

$$|F_Q[\rho; H] - F_Q[\sigma; H]| \leq 32 \sqrt{1 - F^2(\rho, \sigma)} \|H\|^2, \quad (\text{A5})$$

$$|F_Q[\rho; H] - F_Q[\sigma; H]| \leq 32 D_B(\rho, \sigma) \|H\|^2, \quad (\text{A6})$$

and

$$|F_Q[\rho; H] - F_Q[\sigma; H]| \leq 32 \sqrt{\|\rho - \sigma\|_1} \|H\|^2, \quad (\text{A7})$$

where F and D_B stand for the Uhlmann fidelity and the Bures distance respectively.

Proof. The key ingredient of our proof is the fact that the QFI can generally (*not only* for unitary encodings) be expressed as

$$F_Q[\rho_\varphi]|_{\varphi=\varphi_0} = 4 \min_{|\psi_\varphi\rangle} \langle\psi_\varphi|\dot{\psi}_\varphi\rangle \Big|_{\varphi=\varphi_0}, \quad (\text{A8})$$

where $|\dot{\psi}_\varphi\rangle = d|\psi_\varphi\rangle/d\varphi$ and the minimization is in principle performed over all purifications $|\psi_\varphi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E$ of ρ_φ for a given parameter true value φ_0 [17, 44].

It turns out, however, that in this minimization it is enough to consider only a family of purifications of ρ_φ valid at φ_0 given by

$$|\psi_\varphi\rangle = e^{-ih_E(\varphi-\varphi_0)} |\tilde{\psi}_\varphi\rangle, \quad (\text{A9})$$

where $|\tilde{\psi}_\varphi\rangle$ is some fixed purification of ρ_φ and h_E is any Hermitian operator acting on the ancillary subsystem \mathcal{H}_E (notice that h_E is independent of φ) [44]. Moreover, in our case, i.e., when the quantum evolution encoding the parameter φ is unitary, any purification of ρ_φ takes the form

$$|\tilde{\psi}_\varphi\rangle = U_\varphi \otimes \mathbb{1}_E |\psi\rangle, \quad (\text{A10})$$

for some purification $|\psi\rangle$ of ρ .

Now, by substituting Eqs. (A9) and (A10) into Eq. (A8) one obtains an equivalent expression for QFI given by

$$F_Q[\rho; H] = 4 \min_{h_E} \langle\psi|(H + h_E)^2|\psi\rangle, \quad (\text{A11})$$

in which: $H + h_E = H \otimes \mathbb{1}_E + \mathbb{1}_S \otimes h_E$, H is the parameter-encoding Hamiltonian considered (acting on the system), and the minimization is performed over all Hermitian operators h_E acting on the environment. It should be noticed that, in agreement with definition (A4), formula (A11) no longer depends on the parameter φ .

What is more, having the purification-based QFI definition (A11) for a unitary encoding at hand, we may explicitly construct the optimal h_E for a given Hamiltonian H and a state $\rho = \sum_i \lambda_i |\xi_i\rangle\langle\xi_i|$. In particular, we may assume that the fixed purification of ρ appearing in Eq. (A11) is the canonical one generated by the eigensystem of ρ , that is,

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |\xi_i\rangle |i\rangle. \quad (\text{A12})$$

Moreover, denoting by h_{ij} the entries of h_E in the standard basis of \mathcal{H}_E , let us define the following multivariable function

$$\begin{aligned} f(\{h_{ij}\}) &= \langle\psi|(H + h_E)^2|\psi\rangle \\ &= \sum_i \lambda_i \langle\xi_i|H^2|\xi_i\rangle + \sum_{ij} \lambda_i h_{ij} h_{ji} \\ &\quad + 2 \sum_{ij} \sqrt{\lambda_i \lambda_j} \langle\xi_i|H|\xi_j\rangle h_{ij}. \end{aligned} \quad (\text{A13})$$

The necessary condition that this function has a minimum at some h_E is that its derivatives over all h_{ij} vanish at h_E . This gives us the following system of equations

$$2\sqrt{\lambda_i \lambda_j} \langle\xi_i|H|\xi_j\rangle + (\lambda_i + \lambda_j) h_{ji} = 0 \quad (\text{A14})$$

which implies that

$$h_{ij} = -\frac{2\sqrt{\lambda_i\lambda_j}}{\lambda_i + \lambda_j} \langle \xi_j | H | \xi_i \rangle. \quad (\text{A15})$$

The resulting matrix h is clearly Hermitian. Moreover, due to the fact that the function f is convex [which follows from convexity of the square function ([45, p. 113])], the above solution corresponds to its global minimum.

Now, stemming from the QFI definition (A11) and the optimal form of h_E (A15), we prove the first QFI continuity relation (A5). Firstly, let $|\psi_\sigma\rangle$ and h_E^σ be the purification of a given state σ and the corresponding Hamiltonian realising the minimum in Eq. (A11) for this state. Furthermore, let $|\psi_\rho\rangle$ be some, for the time being unspecified, purification of another state ρ . At this point, it should be noticed that both purifications $|\psi_\sigma\rangle$ and $|\psi_\rho\rangle$ can be chosen so that they belong to the same Hilbert space (in other words, the ancillary Hilbert space \mathcal{H}_E can be taken the same for both purifications).

Let us finally assume, without any loss of generality, that $F_Q[\rho; H] \geq F_Q[\sigma; H]$, i.e., ρ is a better state with respect to the metrological task considered. Noting that

$$\begin{aligned} F_Q[\rho; H] &= 4 \min_{h_E} \langle \psi_\rho | (H + h_E)^2 | \psi_\rho \rangle \\ &\leq 4 \langle \psi_\rho | (H + h_E^\sigma)^2 | \psi_\rho \rangle, \end{aligned} \quad (\text{A16})$$

we may upper-bound the QFI difference as

$$\begin{aligned} F_Q[\rho; H] - F_Q[\sigma; H] &\leq 4 [\langle \psi_\rho | (H + h_E^\sigma)^2 | \psi_\rho \rangle \\ &\quad - \langle \psi_\sigma | (H + h_E^\sigma)^2 | \psi_\sigma \rangle] \\ &= 4 \text{Tr}[(\psi_\rho - \psi_\sigma)(H + h_E^\sigma)^2], \end{aligned} \quad (\text{A17})$$

where by ψ_ρ and ψ_σ we denote projectors onto $|\psi_\rho\rangle$ and $|\psi_\sigma\rangle$ respectively. Moreover, exploiting the fact that

$$|\text{Tr}(A^\dagger B)| \leq \|A\|_1 \|B\| \quad (\text{A18})$$

holds for any two operators A and B with $\|\cdot\|$ denoting the matrix norm $\|X\| := \max_{\|\psi\|=1} \|X|\psi\rangle\|$, we arrive at the following expression:

$$F_Q[\rho; H] - F_Q[\sigma; H] \leq 4 \|\psi_\rho - \psi_\sigma\|_1 \|H + h_E^\sigma\|^2. \quad (\text{A19})$$

To obtain a similar relation for the states ρ and σ instead of their purifications, let us notice that for any two pure states $|\psi\rangle$ and $|\phi\rangle$:

$$\|\psi - \phi\|_1 = 2\sqrt{1 - F^2(|\psi\rangle, |\phi\rangle)}, \quad (\text{A20})$$

where F stands for the Uhlmann fidelity (A1). Thus, we may rewrite Eq. (A19) to obtain

$$F_Q[\rho; H] - F_Q[\sigma; H] \leq 8\sqrt{1 - F^2(|\psi_\rho\rangle, |\psi_\sigma\rangle)} \|H + h_E^\sigma\|^2. \quad (\text{A21})$$

Crucially, we can still exploit the freedom in choosing the purification of the state ρ . Concretely, we can choose it to be the one that realises maximum in Eq. (A2), which allows us to just write $F(|\psi_\rho\rangle, |\psi_\sigma\rangle) = F(\rho, \sigma)$. Hence, Eq. (A21) rewrites as

$$F_Q[\rho; H] - F_Q[\sigma; H] \leq 8\sqrt{1 - F^2(\rho, \sigma)} \|H + h_E^\sigma\|^2. \quad (\text{A22})$$

In order to turn the above inequality into the one of Eq. (A5), we need to make Eq. (A22) independent of the auxiliary Hamiltonian h_E^σ . We achieve this by proving that its norm can always be upper-bounded by the norm of the parameter-encoding Hamiltonian, i.e., $\|h_E^\sigma\| \leq \|H\|$.

For this purpose, we recall that h_E^σ that realises the minimum in Eq. (A11) for the state σ (with eigendecomposition $\sigma = \sum_i \mu_i |\eta_i\rangle\langle\eta_i|$) must have the form derived in Eq. (A15). Then, we note that for a Hermitian operator h its operator norm can be expressed as

$$\|h\| := \max_{|\psi\rangle, \|\psi\|=1} |\langle\psi|h|\psi\rangle|. \quad (\text{A23})$$

Let then $|\omega\rangle$ denote the pure state realizing the above maximum for h_E^σ (it is just the eigenvector of h_E^σ corresponding to its eigenvalue with the largest absolute value). Writing $|\omega\rangle$ in the standard basis as $|\omega\rangle = \sum_i \alpha_i |i\rangle$, it follows from Eq. (A15) that the operator norm of h_E^σ is thus given by

$$\|h_E^\sigma\| = \left| \sum_{ij} \alpha_i^* \alpha_j \frac{2\sqrt{\mu_i \mu_j}}{\mu_i + \mu_j} \langle \eta_j | H | \eta_i \rangle \right|. \quad (\text{A24})$$

Now, we introduce the following vector

$$|\eta(t)\rangle = \sum_i \alpha_i^* \sqrt{\mu_i} e^{-t\mu_i} |\eta_i\rangle \quad (\text{A25})$$

with $t \in [0, \infty)$ being some parameter. This vector is normalised so that $\int_0^\infty dt \langle \eta(t) | \eta(t) \rangle = 1/2$. As a result, we may write the operator norm of h_E^σ as follows

$$\|h_E^\sigma\| = 2 \left| \int_0^\infty dt \langle \eta(t) | H | \eta(t) \rangle \right| \quad (\text{A26})$$

and, realising that

$$\begin{aligned} \left| \int_0^\infty dt \langle \eta(t) | H | \eta(t) \rangle \right| &\leq \|H\| \left| \int_0^\infty dt \langle \eta(t) | \eta(t) \rangle \right| \\ &= \frac{1}{2} \|H\|, \end{aligned} \quad (\text{A27})$$

we prove that indeed $\|h_E^\sigma\| \leq \|H\|$. As a result, we may upper-bound the norm appearing in Eq. (A22) as

$$\|H + h_E^\sigma\|^2 \leq (\|H\| + \|h_E^\sigma\|)^2 \leq 4\|H\|^2 \quad (\text{A28})$$

and finally arrive at the first continuity relation (A5).

To prove the second continuity relation (A6), it is enough to notice that

$$\begin{aligned} \sqrt{1 - F^2(\rho, \sigma)} &= \sqrt{[1 - F(\rho, \sigma)][1 + F(\rho, \sigma)]} \quad (\text{A29}) \\ &\leq \sqrt{2[1 - F(\rho, \sigma)]} = D_B(\rho, \sigma), \end{aligned}$$

where to obtain the inequality we have used the fact that $F(\rho, \sigma) \leq 1$ for any pair of states ρ, σ .

Lastly, in order to prove the third continuity relation (A7), we exploit the Fuchs–van de Graaf inequality [28], which states that for any pair of density matrices ρ and σ acting on \mathbb{C}^d

$$1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1. \quad (\text{A30})$$

Hence, it directly follows that $D_B(\rho, \sigma) \leq \sqrt{\|\rho - \sigma\|_1}$ and we obtain the last inequality of Eq. (A7). \square

We now consider the case where the two states ρ and σ are pure. In this situation, the continuity relations of Theorem 1—in particular Eq. (A5)—can be improved by a factor of 3/4, as demonstrated in the following lemma.

Lemma 2. *For any pair of pure states $|\psi\rangle, |\phi\rangle \in \mathcal{H}_S$, and a parameter-encoding Hamiltonian H , the following inequality holds*

$$\begin{aligned} |F_Q[|\psi\rangle; H] - F_Q[|\phi\rangle; H]| &\leq 12\|\psi - \phi\|_1 \|H\|^2 \\ &= 24\sqrt{1 - F^2(|\psi\rangle, |\phi\rangle)} \|H\|^2, \end{aligned} \quad (\text{A31})$$

where ψ and ϕ denote the projectors onto $|\psi\rangle$ and $|\phi\rangle$, respectively.

Proof. We begin by recalling that the QFI for pure states reads

$$F_Q[|\psi\rangle; H] = 4(\langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2). \quad (\text{A32})$$

This allows us to upper-bound the left-hand side of Eq. (A31) as

$$\begin{aligned} |F_Q[|\psi\rangle; H] - F_Q[|\phi\rangle; H]| &\leq 4|\text{Tr}(H^2\psi) - \text{Tr}(H^2\phi)| \\ &\quad + 4|[\text{Tr}(H\psi)]^2 - [\text{Tr}(H\phi)]^2|. \end{aligned} \quad (\text{A33})$$

Let us now concentrate on the second term appearing on the right-hand side of the above inequality. It can be bounded from above as

$$\begin{aligned} |[\text{Tr}(H\psi)]^2 - [\text{Tr}(H\phi)]^2| &= |\text{Tr}(H\psi) - \text{Tr}(H\phi)| \\ &\quad \times |\text{Tr}(H\psi) + \text{Tr}(H\phi)| \\ &\leq 2|\text{Tr}(H\psi) - \text{Tr}(H\phi)| \|H\|, \end{aligned} \quad (\text{A34})$$

where the last inequality is a consequence of the fact that for any normalized $|\psi\rangle$: $\text{Tr}(\psi H) = \langle\psi|H|\psi\rangle \leq \|H\|$. Plugging Eq. (A34) into Eq. (A33), we obtain

$$\begin{aligned} |F_Q[|\psi\rangle; H] - F_Q[|\phi\rangle; H]| &\leq 4|\text{Tr}[(\psi - \phi)H^2]| \\ &\quad + 8|\text{Tr}[(\psi - \phi)H]| \|H\| \end{aligned} \quad (\text{A35})$$

and finally, acknowledging that $|\text{Tr}(X^\dagger Y)| \leq \|X\|_1 \|Y\|$ holds for any two operators X, Y , we arrive at

$$\begin{aligned} |F_Q[|\psi\rangle; H] - F_Q[|\phi\rangle; H]| &\leq 12\|\psi - \phi\|_1 \|H\|^2 \\ &= 24\sqrt{1 - F^2(|\psi\rangle, |\phi\rangle)} \|H\|^2, \end{aligned} \quad (\text{A36})$$

where the last equality stems from Eq. (A20). \square

Although at first sight the inequality (A31) may seem to be less important due the constraint of states purity, it crucially allows us also to tighten the QFI continuity relation (A5) for the case when one of the states considered is pure, while the other can possibly be mixed. In fact, we are able

to do so by using the *convex-roof-based* definition of the QFI that has been introduced for protocols with unitary encoding in Ref. [32]:

Theorem 3. *For any mixed state ρ acting on \mathcal{H}_S and any pure state $|\phi\rangle \in \mathcal{H}_S$ and a given parameter-encoding Hamiltonian H , the following inequality holds true:*

$$|F_Q[\rho; H] - F_Q[|\phi\rangle; H]| \leq 24\sqrt{1 - F^2(\rho, |\phi\rangle)} \|H\|^2. \quad (\text{A37})$$

Proof. Let us first recall that the QFI of a mixed state can be expressed as the convex roof of the variance [32], i.e.,

$$F_Q[\rho; H] = \inf_{\{\lambda_k, |\xi_k\rangle\}} \sum_k p_k F_Q[|\xi_k\rangle; H], \quad (\text{A38})$$

where the infimum is taken over all ensembles $\{p_k, |\xi_k\rangle\}$ such that $\sum_k p_k |\xi_k\rangle\langle\xi_k| = \rho$ (importantly $|\xi_k\rangle$ are normalised but generally *not* orthogonal). Choosing then $\{p_k, |\xi_k\rangle\}$ to be the ensemble realising the minimum in Eq. (A38), one finds that

$$\begin{aligned} |F_Q[\rho; H] - F_Q[|\phi\rangle; H]| &\leq \sum_k p_k |F_Q[|\xi_k\rangle; H] - F_Q[|\phi\rangle; H]| \\ &\leq 24\sqrt{1 - \sum_k p_k F^2(|\xi_k\rangle, |\phi\rangle)} \|H\|^2 \\ &= 24\sqrt{1 - F^2(\rho, |\phi\rangle)} \|H\|^2, \end{aligned} \quad (\text{A39})$$

where the second inequality follows from the pure-states continuity relation (A31) and the concavity of the square root. \square

Lastly, we adopt the above proved Theorems 1 and 3 to the case of the metrology protocol considered in the main text, i.e., the setting when the system investigated consists of N particles, each independently sensing a unitarily encoded parameter of interest. Then, we may always express the overall system Hamiltonian H as a sum of the local ones:

$$H_{\text{loc}} := \sum_{n=1}^N h^{(n)}, \quad (\text{A40})$$

where $h^{(n)}$ represents the parameter-encoding Hamiltonian of the n -th particle and is conveniently normalised so that $\|h^{(n)}\| \leq 1/2$ for all n . In what follows we refer to such Hamiltonians as *local* and denote them by H_{loc} . In particular, as for any H_{loc} , $\|H_{\text{loc}}\|^2 \leq N^2/4$, the three general QFI continuity relations (A5)–(A7) yield the following:

Corollary 4. *For any pair of N -particle states ρ^N and σ^N acting on $(\mathbb{C}^d)^{\otimes N}$, and any local Hamiltonian H_{loc} , the difference in the QFIs of ρ^N and σ^N can always be upper-bounded as:*

$$|F_Q[\rho^N; H_{\text{loc}}] - F_Q[\sigma^N; H_{\text{loc}}]| \leq 8\sqrt{1 - F^2(\rho^N, \sigma^N)} N^2, \quad (\text{A41})$$

$$|F_Q[\rho^N; H_{\text{loc}}] - F_Q[\sigma^N; H_{\text{loc}}]| \leq 8 D_B(\rho^N, \sigma^N) N^2 \quad (\text{A42})$$

and

$$|F_Q[\rho^N; H_{\text{loc}}] - F_Q[\sigma^N; H_{\text{loc}}]| \leq 8 \sqrt{\|\rho^N - \sigma^N\|_1} N^2. \quad (\text{A43})$$

Analogously, we may then also rewrite Theorem 3, which deals with the case of the one of the states being pure.

Corollary 5. *For any pair of N -particle states, a mixed ρ^N and a pure $|\phi^N\rangle$, and a local Hamiltonian H_{loc} , the continuity relation (A37) leads to*

$$|F_Q[\rho^N; H_{\text{loc}}] - F_Q[|\phi^N\rangle; H_{\text{loc}}]| \leq 6 \sqrt{1 - F^2(\rho^N, |\phi^N\rangle)} N^2. \quad (\text{A44})$$

Appendix B: Relating the QFI to geometric measures of entanglement

Here we show that the continuity relation (A5) can be used to link the QFI to a *multiparite entanglement measure*. To this end, we first need to recall the notion of k -producibility.

Consider a multipartite pure state $|\psi^N\rangle \in (\mathbb{C}^d)^{\otimes N}$. We call it k -*producible* with $k \leq N$ if it can be written as [14]

$$|\psi^N\rangle = |\psi_1\rangle \otimes \dots \otimes |\psi_m\rangle, \quad (\text{B1})$$

with each $|\psi_i\rangle$ being a pure state consisting of *at most* k parties. In particular, it follows from this definition that a k - but not $(k-1)$ -producible state contains k particles that are genuinely entangled [20].

This definition can be straightforwardly extended to mixed states: a mixed state ρ^N is k -producible if it is a probabilistic mixture of k -producible pure states. By definition, for every k , the set of all k -producible states \mathcal{S}_k^N is convex. Moreover, such sets of k -producible states form a hierarchy that we schematically depict in Fig. 2. In particular, \mathcal{S}_1^N contains all fully separable states and \mathcal{S}_N^N is the set of all states, and, in general, $\mathcal{S}_1^N \subset \dots \subset \mathcal{S}_N^N$. Note that the set $\mathcal{S}_N^N \setminus \mathcal{S}_{N-1}^N$ thus consists of all N -partite genuinely entangled states.

Exploiting the fact that the sets \mathcal{S}_k^N are convex, one can easily introduce entanglement measures quantifying the extent to which a given N -partite state is non- k -producible. More concretely, for pure states one defines

$$E_k^{\text{prod}}[|\psi^N\rangle] := 1 - \max_{|\phi^N\rangle \in \mathcal{S}_k^N} |\langle \phi^N | \psi^N \rangle|^2, \quad (\text{B2})$$

which is then extended to mixed states by using the convex roof construction, i.e.,

$$E_k^{\text{prod}}[\rho^N] := \inf_{\{p_i, |\psi_i^N\rangle\}} \sum_i p_i E_k^{\text{prod}}[|\psi_i^N\rangle]. \quad (\text{B3})$$

The infimum above is taken over all ensembles $\{p_i, |\psi_i^N\rangle\}$ realising ρ^N , i.e., such that $\sum_i p_i |\psi_i^N\rangle \langle \psi_i^N| = \rho^N$ (importantly $|\psi_i^N\rangle$ are normalised but generally *not* orthogonal).

It is important to note that Eq. (B3) can be rewritten with help of Uhlmann fidelity (A1), (see the appendices of

Ref. [29]), so that the optimisation can be performed over all k -producible mixed states:

$$E_k^{\text{prod}}[\rho^N] = 1 - \max_{\sigma^N \in \mathcal{S}_k^N} F^2(\rho^N, \sigma^N), \quad (\text{B4})$$

what allows us to relate E_k^{prod} to the QFI.

Lastly, let us mention that for the special case of $k=1$ in Eq. (B3), one recovers the definition of the *geometric measure of entanglement* (GME), $E_G[\rho^N] = E_1^{\text{prod}}[\rho^N]$, that we only consider in the main text of this work.

Lemma 6. *For any pure $|\psi^N\rangle \in (\mathbb{C}^d)^{\otimes N}$ and any local Hamiltonian H_{loc} , the following inequality holds:*

$$F_Q[|\psi^N\rangle; H_{\text{loc}}] \leq kN + 6 \sqrt{E_k^{\text{prod}}(|\psi^N\rangle)} N^2. \quad (\text{B5})$$

Proof. Denoting by $|\phi_*^N\rangle$ the k -producible state realizing the maximum in Eq. (B2) for $|\psi^N\rangle$, i.e.,

$$E_G[|\psi^N\rangle] = 1 - |\langle \phi_*^N | \psi^N \rangle|^2, \quad (\text{B6})$$

it follows from Eq. (A44) that

$$F_Q[|\psi^N\rangle] \leq F_Q[|\phi_*^N\rangle] + 6 \sqrt{E_k^{\text{prod}}(|\psi^N\rangle)} N^2. \quad (\text{B7})$$

In order to obtain Eq. (B5) and complete the proof, it remains to utilise the fact that for any k -producible state $\sigma^N \in \mathcal{S}_k^N$, its QFI is upper-bounded as follows [16]:

$$F_Q[\sigma^N] \leq \left\lfloor \frac{N}{k} \right\rfloor k^2 + \left(N - \left\lfloor \frac{N}{k} \right\rfloor k \right)^2 \leq kN. \quad (\text{B8})$$

□

Exploiting the above lemma, we can now prove the following general theorem.

Theorem 7. *For any state ρ^N acting on $(\mathbb{C}^d)^{\otimes N}$ and any local Hamiltonian H_{loc} , the following inequality is true:*

$$F_Q[\rho^N; H_{\text{loc}}] \leq kN + 6 \sqrt{E_k^{\text{prod}}[\rho^N]} N^2, \quad (\text{B9})$$

Proof. Let $\{p_i, |\psi_i^N\rangle\}$ be an ensemble realising ρ^N for which the minimum in Eq. (B3) is achieved. Then, we have the following chain of inequalities

$$\begin{aligned} F_Q[\rho^N; H] &\leq \sum_i p_i F_Q[|\psi_i^N\rangle; H] \\ &\leq kN + 6 \sum_i p_i \sqrt{E_k^{\text{prod}}[|\psi_i^N\rangle]} N^2 \\ &\leq kN + 6 \sqrt{\sum_i p_i E_k^{\text{prod}}[|\psi_i^N\rangle]} N^2 \\ &= kN + 6 \sqrt{E_k^{\text{prod}}[\rho^N]} N^2, \end{aligned} \quad (\text{B10})$$

where the second and the third inequalities follow respectively from Eq. (B5) and the concavity of the square root, while the last equality stems from the definition of E_k^{prod} (B3). □

Remark. For $k = 1$, inequality (B9) relates the QFI for any Hamiltonian of the form (A40), H_{loc} , to the *geometric measure of entanglement* E_G used in the main text:

$$F_Q[\rho^N; H_{\text{loc}}] \leq N + 6\sqrt{E_G[\rho^N]} N^2. \quad (\text{B11})$$

On the other hand, (B9) can be used to derive a lower bound on E_k^{prod} :

$$E_k^{\text{prod}}(\rho^N) \geq \begin{cases} \left(\frac{F_Q[\rho^N; H_{\text{loc}}] - kN}{6N^2} \right)^2, & F_Q[\rho^N; H_{\text{loc}}] > kN \\ 0, & F_Q[\rho^N; H_{\text{loc}}] \leq kN \end{cases} \quad (\text{B12})$$

whose right-hand side scales with N as $(F_Q[\rho^N; H_{\text{loc}}]/6N^2)^2$ in the limit of large N .

The bound (B12) is in general not tight. For instance, for the N -qubit GHZ state

$$|\psi_{\text{GHZ}}^N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}) \quad (\text{B13})$$

the QFI with the Hamiltonian $H_{\text{loc}} = (1/2) \sum_i \sigma_i^z$ amounts to $F_Q[|\psi_{\text{GHZ}}^N\rangle] = N^2$, and hence our bound gives $E_G[|\psi_{\text{GHZ}}^N\rangle] \geq (N^2 - N)^2/36N^4$, which tends to $1/36$ for $N \rightarrow \infty$, while it is known that $E_G[|\psi_{\text{GHZ}}^N\rangle] = 1/2$. Nevertheless, it allows one to lower bound E_k^{prod} for states for which only the QFI is easy to compute. Generally speaking, the bound (B12) provides a non-trivial estimation of E_k^{prod} for all states for which $F_Q[\rho^N; H_{\text{loc}}] > kN$.

Appendix C: Estimating GME for Werner-type states

Let us consider the following class of N -qubit Werner-type states, i.e., a mixture of the GHZ state (B13) and the maximally mixed state:

$$\rho_p^N = p|\psi_{\text{GHZ}}^N\rangle\langle\psi_{\text{GHZ}}^N| + (1-p)\frac{\mathbb{1}_{2^N}}{2^N}. \quad (\text{C1})$$

The GME for these states can be upper bounded as $E_G[\rho_p^N] \leq p/2$. This follows from the facts that E_G is convex and that $E_G[|\psi_{\text{GHZ}}^N\rangle] = 1/2$ for any N . Our aim here is to show that for sufficiently large N this upper bound is very close to the value of $E_G[\rho_p^N]$.

To this end, let us first notice that very recently in Ref. [37] it has been shown that computation of $E_G[\rho_p^N]$ simplifies to the following maximization

$$E_G[\rho_p^N] = \max_{\mu \in [0, \mu_m]} f_p^N(\mu), \quad (\text{C2})$$

where $\mu_m = 2^{N-3}/(2^{N-2} - 1)$ and

$$f_p^N(\mu) = \frac{1}{2} [1 - \mu - \sqrt{\gamma} + 2p\mu + \frac{1-p}{2^N} \left(2\mu + \frac{\mu(\mu + \sqrt{\alpha})}{\mu - 1} \right)] \quad (\text{C3})$$

with $\gamma = (\mu - 1)^2 + 2^{3-N}\mu$ and $\alpha = 1 - \mu + \mu^2$.

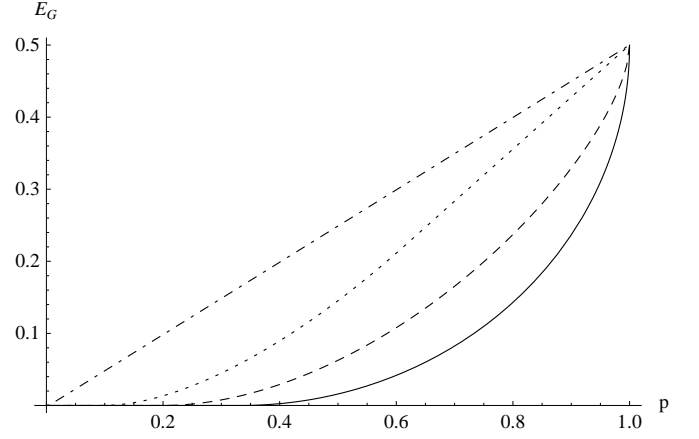


FIG. 4. **Geometric measure of entanglement (GME), E_G , as a function of the parameter p for states given in Eq. (C1).** The number of qubits, N , is chosen to be: 2 (solid line), 3 (dashed line), 4 (dotted line), and 10 (dot-dashed line).

Now, it is clear that $E_G[\rho_p^N] \geq f_p^N(\mu_m)$. It is also not difficult to see that for $N \rightarrow \infty$, $\mu_m \rightarrow 1/2$, $\gamma_m \rightarrow 1/4$ and $\alpha_m \rightarrow 3/4$, where γ_m and α_m are γ and α computed for μ_m . All this implies that $f_p^N(\mu_m) \rightarrow p/2$, and thus $E_G[\rho_p^N] \rightarrow p/2$ for large N .

Furthermore, one should note that the convergence of $E_G[\rho_p^N]$ to $p/2$ with $N \rightarrow \infty$ is quite fast. In other words, already for systems of moderate size ($N = 10$) the upper bound $E_G[\rho_p^N] \leq p/2$ is a good approximation to $E_G[\rho_p^N]$. For this purpose, let us consider the following rough estimation of $|f_p^N(\mu_m) - p/2|$. We first notice that

$$\begin{aligned} \left| f_p^N(\mu_m) - \frac{p}{2} \right| &\leq \frac{1}{2} |1 - \mu_m - \sqrt{\gamma_m}| + \frac{p}{2} |2\mu_m - 1| \\ &\quad + \frac{1-p}{2^N} \left| 2\mu_m + \frac{\mu_m(\mu_m + \sqrt{\alpha_m})}{\mu_m - 1} \right| \end{aligned} \quad (\text{C4})$$

Let us now bound each of the three terms appearing in the above expression. First, we see that

$$|1 - \mu_m - \sqrt{\gamma_m}| \leq \frac{1}{2^{N-2} - 1} + \sqrt{\frac{1}{2^{N-2} - 1}} \quad (\text{C5})$$

Second,

$$|2\mu_m - 1| = \frac{1}{2^{N-2} - 1}, \quad (\text{C6})$$

And finally, for $N \geq 4$,

$$\left| 2\mu_m + \frac{\mu_m(\mu_m + \sqrt{\alpha_m})}{\mu_m - 1} \right| \leq \beta \quad (\text{C7})$$

with $\beta = 4/3 + 2(2 + \sqrt{7})/3 \approx 4.43$. All this gives

$$\left| f_p^N(\mu_m) - \frac{p}{2} \right| \leq \frac{1}{2^{N-2} - 1} + \frac{1}{2} \sqrt{\frac{1}{2^{N-2} - 1}} + \frac{\beta}{2^N}. \quad (\text{C8})$$

One then sees that already for $N = 10$, the difference between then upper bound and the actual value of the GME for ρ_p^N is at most 0.04.

To demonstrate the fast convergence, we have plotted in Fig. 4 the GME, E_G , as a function of the parameter p for N

being: 2, 3, 4, and 10. For large N , the curve becomes almost indistinguishable from $p/2$, which is clear on Fig. 4 already for $N = 10$.

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